

# Bilateral zeta functions and their applications

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## Abstract

We introduce a new type of multiple zeta functions, which we call bilateral zeta functions. We prove that the bilateral zeta function has a nice Fourier series expansion and the Barnes zeta function can be expressed as a finite sum of bilateral zeta functions. By these properties of the bilateral zeta functions, we obtain simple proofs of some formulas, for example, the reflection formula for the multiple gamma function, the inversion formula for the Dedekind  $\eta$ -function, Ramanujan's formula, Fourier expansion of the Barnes zeta function and multiple Iseki's formula.

## 1 Introduction

In the present paper, we introduce a new type of multiple zeta functions(series) analogous to the ones Barnes originally defined in 1904 [Ba], and investigate their fundamental properties. The Barnes zeta functions have been used for the proofs of the basic facts like the transformation formula for theta functions and the reflection formula for multiple gamma functions. Those known proofs are, however, not clear enough, mainly because good explicit expressions of the Barnes zeta functions have not been established. A closer look at this situation tells us that those proofs seem to have paid extra efforts for pursuing the parallelism to the most primitive case. To these difficulties, we will demonstrate how our new type of multiple zeta functions, which we call *bilateral* zeta functions, are useful for the clarification of such zeta-based understanding of the special functions.

Our bilateral zeta function is defined as a periodic function which shares certain basic properties of the Barnes zeta function. Actually, the bilateral zeta function is defined by a sum of two Barnes multiple zeta functions as follows(see Section 4): For  $0 < \arg(\omega_0) \leq \pi$ ,

$$\xi_{r+1}(s, z \mid \omega_0; \boldsymbol{\omega}) = \zeta_{r+1}(s, z + \omega_0 \mid \omega_0, \boldsymbol{\omega}) + \zeta_{r+1}(s, z \mid -\omega_0, \boldsymbol{\omega}).$$

Here  $\zeta_{r+1}(s, z \mid \omega_0, \boldsymbol{\omega})$  ( $\boldsymbol{\omega} := (\omega_1, \dots, \omega_r) \in \mathbf{C}^r$ ) denotes the Barnes multiple zeta function and is given by

$$\zeta_{r+1}(s, z \mid \omega_0, \omega_1, \dots, \omega_r) := \sum_{m_0, m_1, \dots, m_r=0}^{\infty} \frac{1}{(z + m_0\omega_0 + m_1\omega_1 + \dots + m_r\omega_r)^s}.$$

For the absolute convergence of these series, certain conditions on the positions of the parameters and the variable  $z$  should be assumed (see Section 3 and 4 for the precise conditions).

Under those conditions, the domains of absolute convergence in  $s$  for  $\xi_{r+1}$  and  $\zeta_{r+1}$  are  $\operatorname{Re}(s) > r + 1$ .

The bilateral zeta function inherits many of similar properties of the Barnes zeta function. Actually, it can be shown that  $\xi_r$  is continued holomorphically to the whole  $s$ -plane. Moreover, the bilateral zeta function thus defined has obviously the Fourier series expansion which is very nice. Therefore, we easily find that special functions such as multiple  $q$ -shifted factorials are nicely written by the bilateral zeta function. These relationships enable us to derive many important properties of the special functions on the basis of the bilateral zeta functions. Moreover, what is remarkable is that one can show the Barnes zeta function itself turns to be expressed by a finite sum of bilateral zeta functions. In this way, the Fourier expansion of the Barnes zeta functions and multiple Bernoulli polynomials are transparently derived.

The first main result is the following explicit Fourier series expansion (see Theorem 4.7.):

$$\xi_{r+1}(s, z \mid e^{\pi i}; \boldsymbol{\omega}) = \frac{e^{-\frac{\pi}{2}is}(2\pi)^s}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{n^{s-1} e^{2\pi i n z}}{(1 - e^{2\pi i n \omega_1}) \cdots (1 - e^{2\pi i n \omega_r})},$$

for all  $z, \omega_1, \dots, \omega_r \in \mathfrak{H}$  and  $s \in \mathbf{C}$ . Here  $\mathfrak{H}$  is the upper half of the complex plane.

From this Fourier series expansion, small manipulation yields the reflection formula for the multiple gamma function (see corollary 4.10) as

$$\frac{1}{\Gamma_{r+1}(z \mid 1, \boldsymbol{\omega}) \Gamma_{r+1}(1 - z \mid 1, e^{-\pi i} \boldsymbol{\omega})} = \exp \left\{ \frac{(-1)^{r+1} \pi i}{(r+1)!} B_{r+1, r+1}(z \mid 1, \boldsymbol{\omega}) \right\} (x; \mathbf{q})_{r, \infty}.$$

We notice that the formula above has been obtained by Friedman and Ruijsenaars [FR] from Raabe's formula for the integral expression of the Barnes zeta function. The Fourier series expansion of  $\xi_2$  gives also simple proofs of the inversion formula for the Dedekind  $\eta$ -function and Ramanujan's classical formula concerning the special values of the Riemann zeta function (see Proposition 5.3).

The second main result is an expression of the Barnes zeta function by a finite sum of bilateral zeta functions in  $s \in \mathbf{C}$  as follows.

$$\begin{aligned} \zeta_r(s, z \mid \boldsymbol{\omega}) &= \frac{1}{2i \sin(\pi s)} \\ &\cdot \left\{ \sum_{k=1}^r (-1)^{k-1} \xi_r(s, |\boldsymbol{\omega}|_{[1, k-1]}^+ + e^{-\pi i} z \mid \omega_k; \widehat{\boldsymbol{\omega}}^-[k, r](k)) \right. \\ &\quad \left. - \sum_{k=1}^r (-1)^{r-k} e^{-\pi i s} \xi_r(s, z + e^{-\pi i} |\boldsymbol{\omega}|_{[k+1, r]}^+ \mid \omega_k; \widehat{\boldsymbol{\omega}}^-[k, r](k)) \right\} \end{aligned}$$

for  $\omega_1, \dots, \omega_r \in \mathfrak{H}$  and  $z \in D := \{z \in \mathbf{C}^* \mid z = \sum_{k=1}^r a_k \omega_k \ (0 < a_1, \dots, a_r < 1)\}$ . Here we assume that  $\arg(\omega_j) < \arg(\omega_k) \ (1 \leq j < k \leq r)$  (see Section 2 for the definitions of notations).

We remark that the proof of this theorem gives a multiple-analogue of the result in Knopp and Robins [KR]. Further, we obtain also a multiple-analogue of Iseki's formula, a generalization of transformation formula for the theta function [I]. This gives also a generalization of infinite product expressions of the multiple sine functions by Narukawa [Na]. The original proofs of Iseki and that of Narukawa are based on the residues theorem while our proof which follows from Kronecker-type limit formula (4.24) for the bilateral zeta function is much simpler.

Throughout the paper, we denote by  $\mathbf{N}$  be the set of natural numbers,  $\mathbf{Z}$  the ring of rational integers,  $\mathbf{Q}$  the field of rational numbers,  $\mathbf{R}$  the field of real numbers,  $\mathbf{C}$  the field of complex numbers, and put  $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$ ,  $\mathbf{C}^* := \mathbf{C} \setminus \{0\}$ .

## 2 Notations and definitions

For  $c \in \mathbf{C}$ , we always assume that

$$-\pi < \arg c \leq \pi.$$

We set  $\arg 0 := 0$ . We note that  $i := \sqrt{-1} \in \mathfrak{H} = \{z \in \mathbf{C}^* \mid 0 < \arg(z) < \pi\}$ .

For any vector  $\mathbf{X} = (X_1, \dots, X_r) \in \mathbf{C}^r$ , we put

$$(2.1) \quad c\mathbf{X} := (cX_1, \dots, cX_r) \in \mathbf{C}^r \ (c \in \mathbf{C}),$$

$$(2.2) \quad \widehat{\mathbf{X}}(j) := (X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_r) \in \mathbf{C}^{r-1}$$

$$(2.3) \quad = (X_1, \dots, \widehat{X}_j, \dots, X_r),$$

$$\mathbf{X}^-[j] := (X_1, \dots, -X_j, \dots, X_r) \in \mathbf{C}^r,$$

$$(2.4) \quad \mathbf{X}^{-1}[j] := (X_1, \dots, X_j^{-1}, \dots, X_r) \in \mathbf{C}^r,$$

$$(2.5) \quad \mathbf{X}^{-1} := (X_1^{-1}, \dots, X_r^{-1}) \in \mathbf{C}^r,$$

$$(2.6) \quad |\mathbf{X}|^+ := X_1 + \dots + X_r \in \mathbf{C},$$

$$(2.7) \quad |\mathbf{X}|^\times := X_1 \cdots X_r \in \mathbf{C}.$$

When  $m, n, j \in \mathbf{N}$  and  $1 \leq m \leq n \leq r$ , we define

$$(2.8) \quad \mathbf{X}^-[m, n] := (X_1, \dots, X_{m-1}, -X_m, \dots, -X_n, X_{n+1}, \dots, X_r) \in \mathbf{C}^r,$$

$$(2.9) \quad \mathbf{X}^{-1}[m, n] := (X_1, \dots, X_{m-1}, X_m^{-1}, \dots, X_n^{-1}, X_{n+1}, \dots, X_r) \in \mathbf{C}^r,$$

$$(2.10) \quad |\mathbf{X}|_{[m, n]}^+ := \sum_{k=m}^n X_k \in \mathbf{C},$$

$$(2.11) \quad |\mathbf{X}|_{[m, n]}^\times := \prod_{k=m}^n X_k \in \mathbf{C}.$$

$$(2.12) \quad \mathbf{X}^-[r+1, r] := \mathbf{X} \in \mathbf{C}^r,$$

$$(2.13) \quad \mathbf{X}^{-1}[r+1, r] := \mathbf{X} \in \mathbf{C}^r,$$

$$(2.14) \quad |\mathbf{X}|_{[1,0]}^+ := 0,$$

$$(2.15) \quad |\mathbf{X}|_{[r+1,r]}^+ := 0,$$

$$(2.16) \quad |\mathbf{X}|_{[r+1,r]}^\times := 1.$$

Let  $z, \omega_1, \dots, \omega_r \in \mathbf{C}$  and  $x := e^{2\pi iz}$ . Put

$$\boldsymbol{\omega} := (\omega_1, \dots, \omega_r) \in \mathbf{C}^r,$$

$$\mathbf{q} := (q_1, \dots, q_r) \in \mathbf{C}^r,$$

where  $q_k := e^{2\pi i \omega_k}$  ( $k = 1, \dots, r$ ).

**Definition 2.1.** (1) For  $\omega_k \in \mathfrak{H}$  ( $k = 1, \dots, r$ ), we define a  $r$ -ple  $q$ -shifted factorial  $(x; \mathbf{q})_{r,\infty}$  by

$$(2.17) \quad (x; \mathbf{q})_{r,\infty} := \prod_{m_1, \dots, m_r=0}^{\infty} (1 - e^{2\pi i(m_1 \omega_1 + \dots + m_r \omega_r + z)}) = \prod_{m_1, \dots, m_r=0}^{\infty} (1 - q_1^{m_1} \dots q_r^{m_r} x).$$

If  $r = 0$ , we put

$$(2.18) \quad (x)_{0,\infty} := 1 - e^{2\pi iz} = 1 - x.$$

(2) For  $\text{Im}(\omega_1), \dots, \text{Im}(\omega_l) < 0$  and  $\text{Im}(\omega_{l+1}), \dots, \text{Im}(\omega_r) > 0$ , that is,  $|q_1|, \dots, |q_l| > 1$  and  $|q_{l+1}|, \dots, |q_r| < 1$ , we define the generalized  $q$ -shifted factorial  $(\widetilde{x}; \mathbf{q})_{r,\infty}$  by

$$(2.19) \quad (\widetilde{x}; \mathbf{q})_{r,\infty} := (|\mathbf{q}^{-1}|_{[1,l]}^\times x; \mathbf{q}^{-1}[1, l])_{r,\infty}^{(-1)^l} \\ = \prod_{m_1, \dots, m_r=0}^{\infty} (1 - q_1^{-(m_1+1)} \dots q_l^{-(m_l+1)} q_{l+1}^{m_{l+1}} \dots q_r^{m_r} x)^{(-1)^l}.$$

Put  $z_k := z/\omega_k$ ,  $x_k := e^{2\pi i z_k}$ ,  $\omega_{jk} := \omega_j/\omega_k$  and  $q_{jk} := e^{2\pi i \omega_{jk}}$ . Define

$$(2.20) \quad \boldsymbol{\omega}_k := (\omega_{1k}, \dots, \omega_{kk}, \dots, \omega_{rk}),$$

$$(2.21) \quad \mathbf{q}_k := (q_{1k}, \dots, q_{kk}, \dots, q_{rk}),$$

$$(2.22) \quad \widehat{\boldsymbol{\omega}}_k := \widehat{\boldsymbol{\omega}}_k(k) = (\omega_{1k}, \dots, \widehat{\omega}_{kk}, \dots, \omega_{rk}),$$

$$(2.23) \quad \widehat{\mathbf{q}}_k := \widehat{\mathbf{q}}_k(k) = (q_{1k}, \dots, \widehat{q}_{kk}, \dots, q_{rk}).$$

The following lemma is obvious.

**Lemma 2.2.** Assume that  $\omega_1, \dots, \omega_r \in \mathfrak{H}$  satisfy the order condition [ORC] which refers as

$$\arg(\omega_j) < \arg(\omega_k) \quad (j < k).$$

Then, for a fixed  $k$ , one has

$$(2.24) \quad (\widetilde{x_k; \widehat{\mathbf{q}}_k})_{r-1,\infty} = (|\mathbf{q}_k^{-1}|_{[1,k-1]}^\times x_k; \widehat{\mathbf{q}}_k^{-1}[1, k-1])_{r-1,\infty}^{(-1)^{k-1}},$$

$$(2.25) \quad (\widetilde{x_k^{-1}; \widehat{\mathbf{q}}_k^{-1}})_{r-1,\infty} = (|\mathbf{q}_k|_{[k+1,r]}^\times x_k^{-1}; \widehat{\mathbf{q}}_k^{-1}[1, k-1])_{r-1,\infty}^{(-1)^{r-k}}.$$

### 3 Barnes zeta functions

In this section, we recall various properties of the Barnes zeta function from Barnes [Ba]. Let  $r \in \mathbf{N}_0$ . For  $\operatorname{Re}(s) > r + 1$ , we define  $(r + 1)$ -ple Barnes zeta functions  $\zeta_{r+1}(s, z \mid \omega_0, \boldsymbol{\omega})$  by the series

$$(3.1) \quad \zeta_{r+1}(s, z \mid \omega_0, \boldsymbol{\omega}) := \sum_{m_0, \dots, m_r=0}^{\infty} \frac{1}{(z + m_0\omega_0 + m_1\omega_1 + \dots + m_r\omega_r)^s}.$$

Here  $z, \omega_0$  and  $\boldsymbol{\omega}$  satisfy the following one-side condition [OC].

$$\max\{\arg(z), \arg(\omega_0), \arg(\omega_1), \dots, \arg(\omega_r)\} - \min\{\arg(z), \arg(\omega_0), \arg(\omega_1), \dots, \arg(\omega_r)\} < \pi.$$

Throughout this paper, when we consider the Barnes zeta function, we always assume that  $z, \omega_0$  and  $\boldsymbol{\omega}$  satisfy the condition [OC]. For the convenience, we set

$$(3.2) \quad \zeta_0(s, z) := z^{-s}.$$

The Barnes zeta function  $\zeta_{r+1}(s, z \mid \omega_0, \boldsymbol{\omega})$  converges absolutely and uniformly for any compact set in the domain  $\operatorname{Re}(s) > r + 1$ . It is well known that  $\zeta_{r+1}(s, z \mid \omega_0, \boldsymbol{\omega})$  is continued meromorphically to the whole plane  $\mathbf{C}$ .

**Lemma 3.1.** *If  $\alpha \in \mathbf{C}^*$  satisfies the conditions*

$$(3.3) \quad -\pi < \arg(\alpha) + \arg(z) \leq \pi, \quad -\pi < \arg(\alpha) + \arg(\omega_j) \leq \pi \quad (1 \leq j \leq r),$$

*then the following equality holds.*

$$(3.4) \quad \zeta_r(s, \alpha z \mid \alpha \boldsymbol{\omega}) = \alpha^{-s} \zeta_r(s, z \mid \boldsymbol{\omega}).$$

*Proof.* It suffices to show the equation (3.4) when  $\operatorname{Re}(s) > r$ . For  $\alpha \in \mathbf{C}^*$  satisfying the condition, we have

$$\begin{aligned} \zeta_r(s, \alpha z \mid \alpha \boldsymbol{\omega}) &= \sum_{m_1, \dots, m_r=0}^{\infty} \frac{1}{(\alpha z + m_1\alpha\omega_1 + \dots + m_r\alpha\omega_r)^s} \\ &= \sum_{m_1, \dots, m_r=0}^{\infty} \frac{1}{\alpha^s (z + m_1\omega_1 + \dots + m_r\omega_r)^s} = \alpha^{-s} \zeta_r(s, z \mid \boldsymbol{\omega}). \end{aligned}$$

□

**Lemma 3.2.** (1) *We have*

$$(3.5) \quad \zeta_r(s, z + \omega_k \mid \boldsymbol{\omega}) = \zeta_r(s, z \mid \boldsymbol{\omega}) - \zeta_{r-1}(s, z \mid \widehat{\boldsymbol{\omega}}(k)), \quad (k = 1, \dots, r).$$

(2) *Let  $X_r := \{-m_1\omega_1 - \dots - m_r\omega_r \mid m_1, \dots, m_r \in \mathbf{N}_0\}$  and put  $X_0 := \{0\}$ . The function  $\zeta_r(s, z \mid \boldsymbol{\omega})$  is continued analytically to  $\mathbf{C} \setminus X_r$  as a multivalued holomorphic function in  $z$ .*

*Proof.* (1) : The relation follows immediately from the definition.

(2) : By Lemma 3.1, it suffices to show the assertion when  $\operatorname{Re}(\omega_k) > 0 (k = 1, \dots, r)$ . By the relation  $\zeta_1(s, z + 1 \mid 1) = \zeta_1(s, z \mid 1) - z^{-s}$ , we have for all  $n \in \mathbf{N}$

$$\zeta_1(s, z \mid 1) = \zeta_1(s, z + n \mid 1) + \sum_{m=0}^{n-1} (z + m)^{-s}.$$

Hence, one finds that  $\zeta_1(s, z \mid 1)$  is continued analytically to the region  $\{z \in \mathbf{C} \mid \operatorname{Re}(z) > -n\} \setminus X_1$  as a holomorphic function in  $z$ . This proves the case  $r = 1$ .

For  $r \geq 2$ , by (3.5), we have

$$\zeta_r(s, z \mid \boldsymbol{\omega}) = \zeta_r(s, z + n\omega_r \mid \boldsymbol{\omega}) + \sum_{m=0}^{n-1} \zeta_{r-1}(s, z + m\omega_r \mid \widehat{\boldsymbol{\omega}}(r)).$$

By induction, we observe that  $\zeta_r(s, z \mid \boldsymbol{\omega})$  is continued analytically to  $\{z \in \mathbf{C} \mid \operatorname{Re}(z) > -n\operatorname{Re}(\omega_r)\} \setminus X_r$  for all  $n \in \mathbf{N}$ . Hence the desired claim follows.  $\square$

In order to describe special values of the Barnes zeta function, we recall the multiple Bernoulli polynomials [Ba]. We follow the notational convention in [Na].

**Definition 3.3.** We define the multiple Bernoulli polynomials  $B_{r,n}(z \mid \boldsymbol{\omega})$  by a generating function as

$$(3.6) \quad \frac{t^r e^{zt}}{(e^{\omega_1 t} - 1) \cdots (e^{\omega_r t} - 1)} = \sum_{k=0}^{\infty} B_{r,k}(z \mid \boldsymbol{\omega}) \frac{t^k}{k!}.$$

Here  $z, \omega_1, \dots, \omega_r$  do not necessary satisfy the condition [OC].

The multiple Bernoulli polynomial  $B_{r,n}(z \mid \boldsymbol{\omega})$  is actually a polynomial of degree  $n$  in  $z$  and is symmetric in  $\omega_1, \dots, \omega_r$ .

**Example 3.4.**

$$(3.7) \quad B_{r,0}(z \mid \boldsymbol{\omega}) = \frac{1}{|\boldsymbol{\omega}|},$$

$$(3.8) \quad B_{2,2}(z \mid \omega_1, \omega_2) = \frac{1}{\omega_1 \omega_2} z^2 - \frac{\omega_1 + \omega_2}{\omega_1 \omega_2} z + \frac{\omega_1^2 + \omega_2^2 + 3\omega_1 \omega_2}{6\omega_1 \omega_2}.$$

The following lemma is essentially due to Barnes and can be seen in [Na] (see the formulas (12)-(17)).

**Lemma 3.5.** *It holds that*

$$(3.9) \quad B_{r,n}(cz \mid c\boldsymbol{\omega}) = c^{n-r} B_{r,n}(z \mid \boldsymbol{\omega}) \quad (c \in \mathbf{C}^*),$$

$$(3.10) \quad B_{r,n}(|\boldsymbol{\omega}|^+ - z \mid \boldsymbol{\omega}) = (-1)^n B_{r,n}(z \mid \boldsymbol{\omega}),$$

$$(3.11) \quad B_{r,n}(z + \omega_j \mid \boldsymbol{\omega}) - B_{r,n}(z \mid \boldsymbol{\omega}) = n B_{r-1,n-1}(z \mid \widehat{\boldsymbol{\omega}}(j)),$$

$$(3.12) \quad B_{r,n}(z \mid \boldsymbol{\omega}^-[j]) = -B_{r,n}(z + \omega_j \mid \boldsymbol{\omega}),$$

$$(3.13) \quad B_{r,n}(z \mid \boldsymbol{\omega}) + B_{r,n}(z \mid \boldsymbol{\omega}^-[j]) = -n B_{r-1,n-1}(z \mid \widehat{\boldsymbol{\omega}}(j)),$$

$$(3.14) \quad \frac{d}{dz} B_{r,n}(z \mid \boldsymbol{\omega}) = n B_{r,n-1}(z \mid \boldsymbol{\omega}).$$

The special values of the Barnes multiple zeta functions are given by the multiple Bernoulli polynomial as follows.

**Lemma 3.6.** ([Ba]) (1) For all  $m \in \mathbf{N}$ ,

$$(3.15) \quad \zeta_r(1-m, z \mid \boldsymbol{\omega}) = (-1)^r \frac{(m-1)!}{(m+r-1)!} B_{r,r+m-1}(z \mid \boldsymbol{\omega}).$$

(2) For  $m = 1, \dots, r$ ,

$$(3.16) \quad \operatorname{Res}_{s=m} \zeta_r(s, z \mid \boldsymbol{\omega}) ds = \frac{(-1)^{r-m}}{(m-1)!(r-m)!} B_{r,r-m}(z \mid \boldsymbol{\omega}).$$

## 4 Bilateral zeta functions

**Definition 4.1.** Let  $r \in \mathbf{N}_0$ . Assume that  $z, \omega_0$  and  $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)$  satisfy the strong one-side condition [SOC], that is given by

$$\begin{aligned} \min\{\arg(\pm\omega_0)\} &\leq \arg(z) \leq \max\{\arg(\pm\omega_0)\}, \\ \min\{\arg(\pm\omega_0)\} &< \arg(\omega_j) < \max\{\arg(\pm\omega_0)\} \quad (1 \leq j \leq r). \end{aligned}$$

For  $\operatorname{Re}(s) > r+1$ , we define the bilateral  $(r+1)$ -ple zeta function  $\xi_{r+1}$  as follows.

$$(4.1) \quad \xi_{r+1}(s, z \mid \omega_0'; \boldsymbol{\omega}) := \zeta_{r+1}(s, z + \omega_0 \mid \omega_0, \boldsymbol{\omega}) + \zeta_{r+1}(s, z \mid -\omega_0, \boldsymbol{\omega}).$$

Here, we put  $\omega_0' := |\omega_0| e^{i \max\{\arg(\pm\omega_0)\}}$ .

Throughout this paper, we assume that  $0 < \arg(\omega_0) \leq \pi$ . Hence we may write

$$\begin{aligned} \xi_{r+1}(s, z \mid \omega_0'; \boldsymbol{\omega}) &= \zeta_{r+1}(s, z \mid \omega_0; \boldsymbol{\omega}) \\ &= \zeta_{r+1}(s, z + \omega_0 \mid \omega_0, \boldsymbol{\omega}) + \zeta_{r+1}(s, z \mid e^{-\pi i} \omega_0, \boldsymbol{\omega}). \end{aligned}$$

In addition, when we consider the bilateral zeta function, we always assume that  $z, \omega_0, \boldsymbol{\omega}$  satisfy the condition [SOC].

**Lemma 4.2.** (1) The series expression of the bilateral zeta function  $\xi_{r+1}(s, z \mid \omega_0; \boldsymbol{\omega})$  converges absolutely in the domain  $\operatorname{Re}(s) > r+1$ .

(2) It holds that

$$(4.2) \quad \xi_{r+1}(s, z \mid \omega_0; \boldsymbol{\omega}) = \zeta_{r+1}(s, z \mid \omega_0, \boldsymbol{\omega}) + \zeta_{r+1}(s, z + e^{-\pi i} \omega_0 \mid e^{-\pi i} \omega_0, \boldsymbol{\omega})$$

$$(4.3) \quad = \zeta_{r+1}(s, z \mid \omega_0, \boldsymbol{\omega}) + \zeta_{r+1}(s, z \mid e^{-\pi i} \omega_0, \boldsymbol{\omega}) - \zeta_r(s, z \mid \boldsymbol{\omega}).$$

(3) The bilateral zeta function  $\xi_{r+1}(s, z \mid \omega_0; \boldsymbol{\omega})$  is continued analytically to the whole  $s$ -plane  $\mathbf{C}$ . Moreover  $\xi_{r+1}(s, z \mid \omega_0; \boldsymbol{\omega})$  is continued analytically to a multivalued holomorphic function of  $z$  in  $\mathbf{C} \setminus (X_r \cup \{n\omega_0 \mid n \in \mathbf{Z}\})$ .

*Proof.* (1), (2) : These are obvious from the definition and (3.5) of Lemma 3.2.

(3) : The first assertion follows from the relations above and analytic continuation of the Barnes zeta function. By Lemma 3.2, we see that  $\zeta_{r+1}(s, z \mid \omega_0, \boldsymbol{\omega})$ ,  $\zeta_{r+1}(s, z \mid e^{-\pi i} \omega_0, \boldsymbol{\omega})$  and  $\zeta_r(s, z \mid \boldsymbol{\omega})$  are continued analytically to multivalued holomorphic functions of  $z$  in  $\mathbf{C} \setminus (X_r \cup \{-n\omega_0 \mid n \in \mathbf{N}_0\})$ ,  $\mathbf{C} \setminus (X_r \cup \{n\omega_0 \mid n \in \mathbf{N}_0\})$  and  $\mathbf{C} \setminus X_r$  respectively.  $\square$

**Lemma 4.3.** (1) *It holds that*

$$(4.4) \quad \xi_{r+1}(s, z + \omega_0 \mid \omega_0; \boldsymbol{\omega}) = \xi_{r+1}(s, z \mid \omega_0; \boldsymbol{\omega}).$$

(2) *For  $k = 1, \dots, r$ ,*

$$(4.5) \quad \xi_{r+1}(s, z + \omega_k \mid \omega_0; \boldsymbol{\omega}) = \xi_{r+1}(s, z \mid \omega_0; \boldsymbol{\omega}) - \xi_r(s, z \mid \omega_0; \widehat{\boldsymbol{\omega}}(k)).$$

*Proof.* (1) By (4.1) and Lemma 4.2,

$$\begin{aligned} \xi_{r+1}(s, z + \omega_0 \mid \omega_0; \boldsymbol{\omega}) &= \zeta_{r+1}(s, z + \omega_0 \mid \omega_0, \boldsymbol{\omega}) + \zeta_{r+1}(s, (z + \omega_0) + e^{-\pi i} \omega_0 \mid e^{-\pi i} \omega_0, \boldsymbol{\omega}) \\ &= \zeta_{r+1}(s, z + \omega_0 \mid \omega_0, \boldsymbol{\omega}) + \zeta_{r+1}(s, z \mid e^{-\pi i} \omega_0, \boldsymbol{\omega}) \\ &= \xi_{r+1}(s, z \mid \omega_0; \boldsymbol{\omega}). \end{aligned}$$

(2) It is obvious from the definition and Lemma 3.2.  $\square$

**Lemma 4.4.** *If  $\alpha \in \mathbf{C}^*$  satisfies the conditions*

$$(4.6) \quad \begin{aligned} -\pi &< \arg(\alpha) + \arg(z) \leq \pi, \\ 0 &< \arg(\alpha) + \arg(\omega_0) \leq \pi, \\ -\pi &< \arg(\alpha) + \arg(\omega_j) \leq \pi \quad (1 \leq j \leq r), \end{aligned}$$

*then the following equality holds.*

$$(4.7) \quad \xi_{r+1}(s, \alpha z \mid \alpha \omega_0; \alpha \boldsymbol{\omega}) = \alpha^{-s} \xi_{r+1}(s, z \mid \omega_0; \boldsymbol{\omega}).$$

*Proof.* By (3.4) of Lemma 3.1, we obtain

$$\begin{aligned} \xi_{r+1}(s, \alpha z \mid \alpha \omega_0; \alpha \boldsymbol{\omega}) &= \zeta_{r+1}(s, \alpha(z + \omega_0) \mid \alpha \omega_0, \alpha \boldsymbol{\omega}) + \zeta_{r+1}(s, \alpha z \mid \alpha e^{-\pi i} \omega_0, \alpha \boldsymbol{\omega}) \\ &= \alpha^{-s} (\zeta_{r+1}(s, z + \omega_0 \mid \omega_0, \boldsymbol{\omega}) + \zeta_{r+1}(s, z \mid e^{-\pi i} \omega_0, \boldsymbol{\omega})) \\ &= \alpha^{-s} \xi_{r+1}(s, z \mid \omega_0; \boldsymbol{\omega}). \end{aligned}$$

$\square$

**Corollary 4.5.** *One has*

$$(4.8) \quad \xi_{r+1}(s, z \mid \omega_0; \boldsymbol{\omega}) = \left( \frac{e^{\pi i}}{\omega_0} \right)^s \xi_{r+1}(s, e^{\pi i} z_0 \mid e^{\pi i}; e^{\pi i} \boldsymbol{\omega}_0).$$



*Proof.* By the condition [SOC],  $\alpha = \frac{e^{\pi i}}{\omega_0}$  satisfies (4.6) in Lemma 4.4. Therefore, we have

$$\xi_{r+1}(s, e^{\pi i} z_0 \mid e^{\pi i}; e^{\pi i} \omega_0) = \left( \frac{e^{\pi i}}{\omega_0} \right)^{-s} \xi_{r+1}(s, z \mid \omega_0; \omega).$$

□

**Lemma 4.6.** *If  $z \in \mathfrak{H}$ , for all  $s \in \mathbf{C}$ , one has*

$$(4.9) \quad \xi_1(s, z \mid e^{\pi i}) = \frac{e^{-\frac{\pi}{2}is} (2\pi)^s}{\Gamma(s)} \sum_{n=1}^{\infty} n^{s-1} e^{2\pi i n z}.$$

*Proof.* We may assume  $\operatorname{Re}(s) > 1$ . Since  $z \in \mathfrak{H}$ , we observe

$$\begin{aligned} \xi_1(s, z \mid e^{\pi i}) &= \zeta_1(s, z + e^{\pi i} \mid e^{\pi i}) + \zeta_1(s, z \mid 1) \\ &= \sum_{n=0}^{\infty} \frac{1}{(ne^{\pi i} + e^{\pi i} + z)^s} + \sum_{n=0}^{\infty} \frac{1}{(n+z)^s} = \frac{e^{-\frac{\pi}{2}is} (2\pi)^s}{\Gamma(s)} \sum_{n=1}^{\infty} n^{s-1} e^{2\pi i n z}, \end{aligned}$$

where we have used the Lipschitz's formula ([AAR]) for the third equality. □

**Theorem 4.7.** *If  $z, \omega_1, \dots, \omega_r \in \mathfrak{H}$ , for all  $s \in \mathbf{C}$ , one has*

$$(4.10) \quad \xi_{r+1}(s, z \mid e^{\pi i}; \omega) = \frac{e^{-\frac{\pi}{2}is} (2\pi)^s}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{n^{s-1} e^{2\pi i n z}}{(1 - e^{2\pi i n \omega_1}) \dots (1 - e^{2\pi i n \omega_r})}.$$

*Proof.* It is enough to show the assertion when  $\operatorname{Re}(s) > r + 1$ . Since the series for  $\xi_{r+1}(s, z \mid e^{\pi i}; \omega)$  converges absolutely, we observe

$$\begin{aligned} \xi_{r+1}(s, z \mid e^{\pi i}; \omega) &= \sum_{m_1, \dots, m_r=0}^{\infty} \xi_1(s, z + m_1 \omega_1 + \dots + m_r \omega_r \mid e^{\pi i}) \\ &= \sum_{m_1, \dots, m_r=0}^{\infty} \frac{e^{-\frac{\pi}{2}is} (2\pi)^s}{\Gamma(s)} \sum_{n=1}^{\infty} n^{s-1} e^{2\pi i n (z + m_1 \omega_1 + \dots + m_r \omega_r)} \\ &= \frac{e^{-\frac{\pi}{2}is} (2\pi)^s}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{n^{s-1} e^{2\pi i n z}}{(1 - e^{2\pi i n \omega_1}) \dots (1 - e^{2\pi i n \omega_r})}. \end{aligned}$$

Since  $z, \omega_1, \dots, \omega_r \in \mathfrak{H}$ , the second equality follows from (4.9) immediately. □

It is easy to show the following corollary from Lemma 4.5 and Theorem 4.7.

**Corollary 4.8.** (1) *The bilateral zeta function  $\xi_{r+1}(s, z \mid \omega_0; \omega)$  is an entire function in  $s \in \mathbf{C}$ .*

(2) *For all  $m \in \mathbf{N}$ , one has*

$$(4.11) \quad \xi_{r+1}(1 - m, z \mid \omega_0; \omega) = 0.$$

**Corollary 4.9.** *If  $z, \omega_1, \dots, \omega_r \in \mathfrak{H}$ , for all  $m \in \mathbf{N}$ , one has*

$$(4.12) \quad \frac{\partial \xi_{r+1}}{\partial s}(1-m, z \mid e^{\pi i}; \boldsymbol{\omega}) = \frac{(m-1)!}{(2\pi i)^{m-1}} \sum_{n=1}^{\infty} \frac{e^{2\pi i n z}}{n^m (1 - e^{2\pi i n \omega_1}) \dots (1 - e^{2\pi i n \omega_r})}.$$

*In particular,*

$$(4.13) \quad \exp \left( -\frac{\partial \xi_{r+1}}{\partial s}(0, z \mid e^{\pi i}; \boldsymbol{\omega}) \right) = (x; \mathbf{q})_{r, \infty}.$$

*Proof.* If  $z, \omega_1, \dots, \omega_r \in \mathfrak{H}$ , then

$$\begin{aligned} \frac{\partial \xi_{r+1}}{\partial s}(1-m, z \mid e^{\pi i}; \boldsymbol{\omega}) &= \lim_{s \rightarrow 0} \frac{\xi_{r+1}(s+1-m, z \mid e^{\pi i}; \boldsymbol{\omega}) - \xi_{r+1}(1-m, z \mid e^{\pi i}; \boldsymbol{\omega})}{s} \\ &= \lim_{s \rightarrow 0} \frac{\xi_{r+1}(s+1-m, z \mid e^{\pi i}; \boldsymbol{\omega})}{s} \\ &= \frac{(m-1)!}{(2\pi i)^{m-1}} \sum_{n=1}^{\infty} \frac{e^{2\pi i n z}}{n^m (1 - e^{2\pi i n \omega_1}) \dots (1 - e^{2\pi i n \omega_r})}. \end{aligned}$$

The second equality follows from (4.11) and the third one from (4.10). Moreover, by analytic continuation, it is enough to show the assertion (4.13) when  $z \in \mathfrak{H}$ .

$$\begin{aligned} \exp \left( -\frac{\partial \xi_{r+1}}{\partial s}(0, z \mid e^{\pi i}; \boldsymbol{\omega}) \right) &= \exp \left( -\sum_{n=1}^{\infty} \frac{e^{2\pi i n z}}{n(1 - e^{2\pi i n \omega_1}) \dots (1 - e^{2\pi i n \omega_r})} \right) \\ &= \exp \left( -\sum_{n=1}^{\infty} \sum_{m_1, \dots, m_r=0}^{\infty} \frac{e^{2\pi i n(z+m_1\omega_1+\dots+m_r\omega_r)}}{n} \right) \\ &= \exp \left( \log \prod_{m_1, \dots, m_r=0}^{\infty} (1 - e^{2\pi i(m_1\omega_1+\dots+m_r\omega_r+z)}) \right) \\ &= (x; \mathbf{q})_{r, \infty}. \end{aligned}$$

This proves (4.13). □

**Corollary 4.10.** ([FR]) *If  $\omega_1, \dots, \omega_r \in \mathfrak{H}$ , then*

$$(4.14) \quad \frac{1}{\Gamma_{r+1}(z \mid 1, \boldsymbol{\omega}) \Gamma_{r+1}(1-z \mid 1, e^{-\pi i} \boldsymbol{\omega})} = \exp \left\{ \frac{(-1)^{r+1} \pi i}{(r+1)!} B_{r+1, r+1}(z \mid 1, \boldsymbol{\omega}) \right\} (x; \mathbf{q})_{r, \infty}.$$

Here  $\Gamma_r(z \mid \boldsymbol{\omega})$  is the multiple gamma function defined by

$$(4.15) \quad \Gamma_r(z \mid \boldsymbol{\omega}) := \exp \left( \frac{\partial \zeta_r}{\partial s}(0, z \mid \boldsymbol{\omega}) \right).$$

*Proof.* Suppose  $\omega_1, \dots, \omega_r \in \mathfrak{H}$ . Since  $\Gamma_r(z \mid \omega)^{-1}$  and  $\Gamma_{r+1}(1-z \mid 1, e^{-\pi i} \omega)^{-1}$  are continued holomorphically to the whole  $z$ -plane, it is enough to show the assertion when  $z \in \mathfrak{H}$ . By Corollary 4.9, if  $z, \omega_1, \dots, \omega_r \in \mathfrak{H}$ ,

$$(x; \mathbf{q})_{r,\infty} = \exp \left( -\frac{\partial \xi_{r+1}}{\partial s}(0, z \mid e^{\pi i}; \omega) \right).$$

By (4.1), we observe

$$\begin{aligned} \xi_{r+1}(s, z \mid e^{\pi i}; \omega) &= \zeta_{r+1}(s, z \mid 1, \omega) + \zeta_{r+1}(s, z + e^{\pi i} \mid e^{\pi i}, \omega) \\ &= \zeta_{r+1}(s, z \mid 1, \omega) + e^{-\pi i s} \zeta_{r+1}(s, e^{-\pi i} z + 1 \mid 1, e^{-\pi i} \omega). \end{aligned}$$

Therefore, we have

$$\begin{aligned} (x; \mathbf{q})_{r,\infty} &= \exp(\pi i \zeta_{r+1}(0, e^{-\pi i} z + 1 \mid 1, e^{-\pi i} \omega)) \\ &\quad \cdot \exp \left( -\frac{\partial \zeta_{r+1}}{\partial s}(0, z \mid 1, \omega) - \frac{\partial \zeta_{r+1}}{\partial s}(0, e^{-\pi i} z + 1 \mid 1, e^{-\pi i} \omega) \right) \\ &= \exp \left( \frac{-(-1)^{r+1} \pi i}{(r+1)!} B_{r+1,r+1}(z \mid 1, \omega) \right) \frac{1}{\Gamma_{r+1}(z \mid 1, \omega) \Gamma_{r+1}(1-z \mid 1, e^{-\pi i} \omega)}. \end{aligned}$$

The last equality follows from (3.15) and Lemma 3.5 (3.10), (3.12).  $\square$

Now we give the main theorem of this paper.

**Theorem 4.11.** *Suppose that  $r \geq 2$ . Assume that  $\omega_1, \dots, \omega_r \in \mathfrak{H}$  satisfy the condition [ORC]. Put*

$$\begin{aligned} D &:= \left\{ z \in \mathbf{C}^* \mid z = \sum_{k=1}^r a_k \omega_k \ (0 < a_1, \dots, a_r < 1) \right\}, \\ D_+ &:= \{ z \in \mathbf{C}^* \mid \arg(\omega_r) < \arg(z) < \pi \}, \\ D_- &:= \{ z \in \mathbf{C}^* \mid 0 < \arg(z) < \arg(\omega_1) \}, \\ f_+(s, z \mid \omega) &:= \zeta_r(s, e^{-\pi i} z \mid e^{-\pi i} \omega) + (-1)^{r-1} \zeta_r(s, |\omega|^+ + e^{-\pi i} z \mid \omega), \\ f_-(s, z \mid \omega) &:= \zeta_r(s, z \mid \omega) + (-1)^{r-1} \zeta_r(s, z + e^{-\pi i} |\omega|^+ \mid e^{-\pi i} \omega). \end{aligned}$$

(1) *If  $z \in D \cup D_+$ , one has for all  $s \in \mathbf{C}$ ,*

$$(4.16) \quad f_+(s, z \mid \omega) = \sum_{k=1}^r (-1)^{k-1} \xi_r(s, |\omega|_{[1,k-1]}^+ + e^{-\pi i} z \mid \omega_k; \widehat{\omega}^-[k, r](k))$$

$$(4.17) \quad = \frac{e^{\frac{\pi}{2} i s} (2\pi)^s}{\Gamma(s)} \sum_{k=1}^r \omega_k^{-s} \sum_{n=1}^{\infty} n^{s-1} e^{2\pi i n z_k} \prod_{j=1, j \neq k}^r (1 - e^{2\pi i n \omega_{jk}})^{-1}.$$

(2) If  $z \in D \cup D_-$ , one has for all  $s \in \mathbf{C}$ ,

$$(4.18) \quad f_-(s, z \mid \omega) = \sum_{k=1}^r (-1)^{r-k} \xi_r(s, z + e^{-\pi i} |\omega|_{[k+1, r]}^+ \mid \omega_k; \widehat{\omega}^-[k, r](k))$$

$$(4.19) \quad = \frac{e^{\frac{\pi}{2}is} (2\pi)^s}{\Gamma(s)} \sum_{k=1}^r \omega_k^{-s} \sum_{n=1}^{\infty} n^{s-1} e^{-2\pi i n z_k} \prod_{j=1, j \neq k}^r (1 - e^{-2\pi i n \omega_{jk}})^{-1},$$

To prove the main theorem, we need the following lemma.

**Lemma 4.12.** *Suppose that  $\omega_1, \dots, \omega_r$  satisfy the condition [ORC].*

(1) If  $z \in D \cup D_+$ , then

$$(4.20) \quad z_k + e^{\pi i} |\omega_k|_{[1, k-1]}^+ \in \mathfrak{H} \quad (k = 1, \dots, r).$$

(2) If  $z \in D \cup D_-$ , then

$$(4.21) \quad e^{\pi i} z_k + |\omega_k|_{[k+1, r]}^+ \in \mathfrak{H} \quad (k = 1, \dots, r).$$

*Proof.* (1) : The condition [ORC] shows

$$e^{\pi i} \omega_{1k}, \dots, e^{\pi i} \omega_{k-1k}, \omega_{k+1k}, \dots, \omega_{rk} \in \mathfrak{H}.$$

Therefore we have  $e^{\pi i} |\omega_k|_{[1, k-1]}^+ \in \mathfrak{H}$ . When  $z \in D_+$ , the result follows from the definition of  $D_+$ . Let  $z = \sum_{l=1}^r a_l \omega_l$  ( $0 < a_1, \dots, a_r < 1$ )  $\in D$ , then

$$\begin{aligned} z_k + e^{\pi i} |\omega_k|_{[1, k-1]}^+ &= \sum_{l=1}^r a_l \omega_{lk} + e^{\pi i} |\omega_k|_{[1, k-1]}^+ \\ &= \sum_{l=1}^{k-1} (1 - a_l) e^{\pi i} \omega_{lk} + a_k + \sum_{l=k+1}^r a_l \omega_{lk}. \end{aligned}$$

Since  $1 - a_l > 0$  for all  $l = 1, \dots, k-1$ , we obtain (4.20).

(2) : By the condition [ORC], we have  $|\omega_k|_{[k+1, r]}^+ \in \mathfrak{H}$ . Hence,  $z \in D_-$ , the result is proved by a similar argument as in (1). If  $z \in D$ , then

$$\begin{aligned} e^{\pi i} z_k + |\omega_k|_{[k+1, r]}^+ &= \sum_{l=1}^r a_l e^{\pi i} \omega_{lk} + |\omega_k|_{[k+1, r]}^+ \\ &= \sum_{l=1}^{k-1} a_l e^{\pi i} \omega_{lk} + e^{\pi i} a_k + \sum_{l=k+1}^r (1 - a_l) \omega_{lk}. \end{aligned}$$

Since  $1 - a_l > 0$  for all  $l = k+1, \dots, r$ , we obtain (4.21). □

**Proof of Theorem 4.11.** Suppose that  $z \in D \cup D_+$ . Then we have

$$\begin{aligned}
f_+(s, z \mid \omega) &= \sum_{k=0}^{r-1} (-1)^k \{ \zeta_r(s, |\omega|_{[1,k]}^+ + e^{-\pi i} z \mid \omega^-[k+1, r]) \\
&\quad + \zeta_r(s, |\omega|_{[1,k+1]}^+ + e^{-\pi i} z \mid \omega^-[k+2, r]) \} \\
&= \sum_{k=1}^r (-1)^{k-1} \xi_r(s, |\omega|_{[1,k-1]}^+ + e^{-\pi i} z \mid \omega_k; \widehat{\omega}^-[k, r](k)) \\
&= \sum_{k=1}^r (-1)^{k-1} \left( \frac{e^{\pi i}}{\omega_k} \right)^s \xi_r(s, z_k + e^{\pi i} |\omega_k|_{[1,k-1]}^+ \mid e^{\pi i}; e^{\pi i} \widehat{\omega}_k^-[k, r]) \\
&= \sum_{k=1}^r (-1)^{k-1} \left( \frac{e^{\pi i}}{\omega_k} \right)^s \xi_r(s, z_k + e^{\pi i} |\omega_k|_{[1,k-1]}^+ \mid e^{\pi i}; \widehat{\omega}_k^-[1, k-1]).
\end{aligned}$$

The second equality follows from (4.1) and the third one from Lemma 4.4. We remark that all  $r$ -ple zeta functions and bilateral  $r$ -ple zeta functions appeared above are well-defined, since all parameters appeared in the above equalities satisfy the condition [SOC].

In addition, by  $z \in D \cup D_+$ , we may apply Lemma 4.12 to (4.16). Hence it follows that all variables of the bilateral  $r$ -ple zeta functions appeared in (4.16) are in the upper half plane. Therefore, by Theorem 4.7,

$$\begin{aligned}
f_+(s, z \mid \omega) &= \frac{e^{\frac{\pi}{2}is}(2\pi)^s}{\Gamma(s)} \sum_{k=1}^r (-1)^{k-1} \omega_k^{-s} \sum_{n=1}^{\infty} n^{s-1} e^{2\pi i n(z_k + e^{\pi i} |\omega_k|_{[1,k-1]}^+)} \\
&\quad \cdot \prod_{j=1}^{k-1} (1 - e^{-2\pi i n \omega_{jk}})^{-1} \prod_{j=k+1}^r (1 - e^{2\pi i n \omega_{jk}})^{-1} \\
&= \frac{e^{\frac{\pi}{2}is}(2\pi)^s}{\Gamma(s)} \sum_{k=1}^r \omega_k^{-s} \sum_{n=1}^{\infty} n^{s-1} e^{2\pi i n z_k} \prod_{j=1, j \neq k}^r (1 - e^{2\pi i n \omega_{jk}})^{-1}.
\end{aligned}$$

Let  $z \in D \cup D_-$ . Similarly to the discussion above, one observes

$$\begin{aligned}
f_-(s, z \mid \omega) &= \sum_{k=0}^{r-1} (-1)^k \{ \zeta_r(s, z + e^{-\pi i} |\omega|_{[r+1-k, r]}^+ \mid \omega^-[r+1-k, r]) \\
&\quad + \zeta_r(s, z + e^{-\pi i} |\omega|_{[r-k, r]}^+ \mid \omega^-[r-k, r]) \} \\
&= \sum_{k=1}^r (-1)^{r-k} \xi_r(s, z + e^{-\pi i} |\omega|_{[k+1, r]}^+ \mid \omega_k; \widehat{\omega}^-[k, r](k)) \\
&= \sum_{k=1}^r (-1)^{r-k} \left( \frac{e^{\pi i}}{\omega_k} \right)^s \xi_r(s, e^{\pi i} z_k + |\omega_k|_{[k+1, r]}^+ \mid e^{\pi i}; \widehat{\omega}_k^-[1, k-1]) \\
&= \frac{e^{\frac{\pi}{2}is}(2\pi)^s}{\Gamma(s)} \sum_{k=1}^r \omega_k^{-s} \sum_{n=1}^{\infty} n^{s-1} e^{-2\pi i n z_k} \prod_{j=1, j \neq k}^r (1 - e^{-2\pi i n \omega_{jk}})^{-1}.
\end{aligned}$$

This completes the proof of Theorem 4.11.

**Remark 4.13.** Let  $r = 1$  and  $\omega_1 \in \mathfrak{H}$ . Then, for all  $s \in \mathbf{C}$ , we have

$$(4.22) \quad f_+(s, z \mid \omega_1) = \xi_1(s, e^{-\pi i} z \mid \omega_1) = \frac{e^{\frac{\pi}{2}is}(2\pi)^s}{\Gamma(s)} \omega_1^{-s} \sum_{n=1}^{\infty} n^{s-1} e^{2\pi i n z_1} \quad (z \in D_+),$$

and

$$(4.23) \quad f_-(s, z \mid \omega_1) = \xi_1(s, z \mid \omega_1) = \frac{e^{\frac{\pi}{2}is}(2\pi)^s}{\Gamma(s)} \omega_1^{-s} \sum_{n=1}^{\infty} n^{s-1} e^{-2\pi i n z_1} \quad (z \in D_-).$$

In particular, when  $\operatorname{Re}(s) < 0$ , we have (4.22) and (4.23) under the condition  $z \in D$ .

**Lemma 4.14.** Assume that  $z \in D \cup D_{\pm}$ . If  $\arg(\omega_j) \neq \arg(\omega_k)$  ( $j \neq k$ ), then the right-hand sides of (4.17) and (4.19) converges absolutely.

*Proof.* If  $\arg(\omega_j) \neq \arg(\omega_k)$  ( $j \neq k$ ), we may change the order of parameters such as  $\arg(\omega_j) < \arg(\omega_k)$  ( $j < k$ ). Hence, by Lemma 4.12, we obtain the assertion.  $\square$

**Corollary 4.15.** Let  $r \geq 2$ ,  $z \in D$ . Suppose that  $z, \omega_1, \dots, \omega_r \in \mathfrak{H}$  satisfy the condition [ORC]. If  $z \in D \cup D_{\pm}$ , one has

$$(4.24) \quad \frac{\partial f_{\pm}}{\partial s}(1-m, z \mid \omega) = \frac{(-1)^{m-1}(m-1)!}{(2\pi i)^{m-1}} \sum_{k=1}^r \omega_k^{m-1} \sum_{n=1}^{\infty} \frac{e^{\pm 2\pi i n z_k}}{n^m} \prod_{j=1, j \neq k}^r (1 - e^{\pm 2\pi i n \omega_{jk}})^{-1}.$$

In particular, for any  $z \in \mathbf{C}$ ,

$$(4.25) \quad \exp\left(-\frac{\partial f_{\pm}}{\partial s}(0, z \mid \omega)\right) = \prod_{k=1}^r \widetilde{(x_k^{\pm 1}; \widehat{\mathbf{q}}_k^{\pm 1})}_{r-1, \infty}.$$

*Proof.* By (4.16) and Corollary 4.8, one has

$$\begin{aligned} \frac{\partial f_{\pm}}{\partial s}(1-m, z \mid \omega) &= \lim_{s \rightarrow 0} \frac{f_{\pm}(s+1-m, z \mid \omega) - f_{\pm}(1-m, z \mid \omega)}{s} \\ &= \frac{(-1)^{m-1}(m-1)!}{(2\pi i)^{m-1}} \sum_{k=1}^r \omega_k^{m-1} \sum_{n=1}^{\infty} \frac{e^{\pm 2\pi i n z_k}}{n^m} \prod_{j=1, j \neq k}^r (1 - e^{\pm 2\pi i n \omega_{jk}})^{-1}. \end{aligned}$$

Similarly, one obtains

$$\begin{aligned} \exp\left(-\frac{\partial f_{\pm}}{\partial s}(0, z \mid \omega)\right) &= \prod_{k=1}^r \exp\left(-(-1)^{k-1} \frac{\partial \xi_r}{\partial s}(0, z_k + e^{\pi i} |\omega_k|_{[1, k-1]}^+ \mid e^{\pi i}; \widehat{\omega}_k^-[1, k-1])\right) \\ &= \prod_{k=1}^r (|\mathbf{q}_k^{-1}|_{[1, k-1]}^{\times} x_k; \widehat{\mathbf{q}}_k^{-1}[1, k-1])_{r-1, \infty}^{(-1)^{k-1}} \\ &= \prod_{k=1}^r \widetilde{(x_k; \widehat{\mathbf{q}}_k)}_{r-1, \infty}. \end{aligned}$$

The second equality follows from (4.13) of Corollary 4.9 and the third one from (2.24).

Similarly,

$$\begin{aligned} \exp\left(-\frac{\partial f_-}{\partial s}(0, z \mid \omega)\right) &= \prod_{k=1}^r \exp\left(-(-1)^{r-k} \frac{\partial \xi_r}{\partial s}(0, e^{\pi i} z_k + |\omega_k|_{[k+1, r]}^+ \mid e^{\pi i}; \widehat{\omega}_k^-[1, k-1])\right) \\ &= \prod_{k=1}^r (|\mathbf{q}_k|_{[k+1, r]}^\times x_k^{-1}; \widehat{\mathbf{q}}_k^{-1}[1, k-1])_{r-1, \infty}^{(-1)^{r-k}} \\ &= \prod_{k=1}^r (\widetilde{x_k^{-1}}; \widehat{\mathbf{q}}_k^{-1})_{r-1, \infty}. \end{aligned}$$

□

**Remark 4.16.** By Lemma 4.14, if  $z \in D$  and  $\arg(\omega_j) \neq \arg(\omega_k)$  ( $j \neq k$ ), then the right-hand sides of (4.24) and (4.25) both converge absolutely.

## 5 Applications

### 5.1 Dedekind's $\eta$ -inversion formula and Ramanujan's formula

In this subsection, we assume that  $\tau \in \mathfrak{H}$ .

**Lemma 5.1.** (1) For any  $s \in \mathbf{C}$ ,

$$(5.1) \quad \zeta_2(s, \omega_1 \mid \omega_1, \omega_2) - \zeta_2(s, \omega_2 \mid \omega_1, \omega_2) = (\omega_1^{-s} - \omega_2^{-s})\zeta(s).$$

Here  $\zeta(s)$  is the Riemann zeta function.

(2) For any  $N \in \mathbf{N}$ ,

$$(5.2) \quad \lim_{s \rightarrow 2N} (1 - e^{\pi i s}) \zeta_2(s, z \mid \omega_1, \omega_2) = -\frac{\pi i}{\omega_1 \omega_2} \delta_{1, N}.$$

Here  $\delta_{1, N}$  is the Kronecker's delta.

*Proof.* (1) : By Lemma 3.2, we obtain

$$\begin{aligned} (LHS) &= \lim_{z \rightarrow 0} \{\zeta_2(s, z + \omega_1 \mid \omega_1, \omega_2) - \zeta_2(s, z + \omega_2 \mid \omega_1, \omega_2)\} \\ &= \lim_{z \rightarrow 0} \{\{\zeta_2(s, z \mid \omega_1, \omega_2) - \zeta_1(s, z \mid \omega_2)\} - \{\zeta_2(s, z \mid \omega_1, \omega_2) - \zeta_1(s, z \mid \omega_1)\}\} \\ &= \lim_{z \rightarrow 0} \{\{\zeta_1(s, z \mid \omega_1) - z^{-s}\} - \{\zeta_1(s, z \mid \omega_2) - z^{-s}\}\} \\ &= \lim_{z \rightarrow 0} \{\zeta_1(s, z + \omega_1 \mid \omega_1) - \zeta_1(s, z + \omega_2 \mid \omega_2)\} \\ &= \zeta_1(s, \omega_1 \mid \omega_1) - \zeta_1(s, \omega_2 \mid \omega_2) = (\omega_1^{-s} - \omega_2^{-s})\zeta(s). \end{aligned}$$

(2) : By (3.7) and (3.16), we obtain

$$\begin{aligned} (LHS) &= \lim_{s \rightarrow 2N} \frac{1 - e^{\pi i s}}{s - 2N} (s - 2N) \zeta_2(s, z \mid \omega_1, \omega_2) \\ &= -\pi i \operatorname{Res}_{s=2N} \zeta_2(s, z \mid \omega_1, \omega_2) ds = -\pi i B_{2,0}(z \mid \omega_1, \omega_2) \delta_{1,N} = -\frac{\pi i}{\omega_1 \omega_2} \delta_{1,N}. \end{aligned}$$

□

**Lemma 5.2.** *Define*

$$\begin{aligned} g(s, \tau) &:= \zeta_2(s, \tau \mid e^{\pi i}, \tau) - \left( e^{\pi i} \frac{1}{\tau} \right)^s \zeta_2 \left( s, e^{\pi i} \frac{1}{\tau} \mid e^{\pi i}; e^{\pi i} \frac{1}{\tau} \right) \\ &= \zeta_2(s, \tau \mid e^{\pi i}, \tau) - \zeta_2(s, 1 \mid \tau, 1). \end{aligned}$$

(1) *We have*

$$(5.3) \quad \frac{\partial g}{\partial s}(0, \tau) = -\frac{\pi i}{4} + \frac{\pi i}{12} \left( \tau + \frac{1}{\tau} \right) + \frac{1}{2} \log \tau.$$

(2) *For any  $N \in \mathbf{N}$ ,*

$$(5.4) \quad \frac{\partial g}{\partial s}(-2N, \tau) = \frac{\pi i B_{2,2+2N}(0 \mid 1, \tau)}{(2N+2)(2N+1)} + \frac{(-1)^N}{2} (\tau^{2N} - 1)(2N)!(2\pi)^{-2N} \zeta(2N+1).$$

(3) *For any  $N \in \mathbf{N}$ ,*

$$(5.5) \quad g(2N, \tau) = (\tau^{-2N} - 1) \left( -\frac{1}{2} \frac{B_{2N}}{(2N)!} (2\pi i)^{2N} \right) + \frac{\pi i}{\tau} \delta_{1,N}.$$

*Proof.* By (4.1), we observe that

$$\begin{aligned} g(s, \tau) &= \{\zeta_2(s, 1 + \tau \mid 1, \tau) + \zeta_2(s, \tau \mid e^{\pi i}, \tau)\} - \{\zeta_2(s, 1 + \tau \mid \tau, 1) + \zeta_2(s, 1 \mid e^{-\pi i} \tau, 1)\} \\ &= \zeta_2(s, \tau \mid e^{\pi i}, \tau) - \zeta_2(s, 1 \mid e^{-\pi i} \tau, 1) = \zeta_2(s, \tau \mid e^{\pi i}, \tau) - e^{\pi i s} \zeta_2(s, e^{\pi i} \mid \tau, e^{\pi i}). \end{aligned}$$

(1) : Using the above relation, we have

$$\frac{\partial g}{\partial s}(0, \tau) = -\pi i \zeta_2(0, e^{\pi i} \mid \tau, e^{\pi i}) + \frac{\partial}{\partial s} \{\zeta_2(s, \tau \mid e^{\pi i}, \tau) - \zeta_2(s, e^{\pi i} \mid \tau, e^{\pi i})\} \Big|_{s=0}.$$

By (3.15) and (3.8)

$$-\pi i \zeta_2(0, e^{\pi i} \mid \tau, e^{\pi i}) = -\pi i \frac{B_{2,2}(e^{\pi i} \mid \tau, e^{\pi i})}{2!} = \frac{\pi i}{4} + \frac{\pi i}{12} \left( \tau + \frac{1}{\tau} \right).$$

On the other hand, by (5.1) and the fact  $\zeta(0) = -\frac{1}{2}$ , one finds

$$\begin{aligned} &\frac{\partial}{\partial s} \{\zeta_2(s, \tau \mid e^{\pi i}, \tau) - \zeta_2(s, e^{\pi i} \mid \tau, e^{\pi i})\} \Big|_{s=0} \\ &= \frac{\partial}{\partial s} \{(\tau^{-s} - e^{-\pi i s}) \zeta(s)\} \Big|_{s=0} = (-\log \tau + \pi i) \zeta(0) = \frac{1}{2} \log \tau - \frac{\pi i}{2}. \end{aligned}$$



Consequently we obtain

$$\frac{\partial g}{\partial s}(0, \tau) = -\frac{\pi i}{4} + \frac{\pi i}{12} \left( \tau + \frac{1}{\tau} \right) + \frac{1}{2} \log \tau.$$

(2) : Similarly, we have

$$\frac{\partial g}{\partial s}(-2N, \tau) = -\pi i \zeta_2(-2N, e^{\pi i} \mid \tau, e^{\pi i}) + \frac{\partial}{\partial s} \{ \zeta_2(s, \tau \mid e^{\pi i}, \tau) - \zeta_2(s, e^{\pi i} \mid \tau, e^{\pi i}) \} \Big|_{s=-2N}.$$

By (3.15) and Lemma 3.5, we have

$$-\pi i \zeta_2(-2N, e^{\pi i} \mid \tau, e^{\pi i}) = -\pi i \frac{B_{2,2+2N}(e^{\pi i} \mid \tau, e^{\pi i})}{(2N+2)(2N+1)} = \frac{\pi i B_{2,2+2N}(0 \mid 1, \tau)}{(2N+2)(2N+1)}.$$

Also, by the fact  $\zeta(-2N) = 0$  for any  $N \in \mathbf{N}$ , we have

$$\begin{aligned} \frac{\partial}{\partial s} \{ \zeta_2(s, \tau \mid e^{\pi i}, \tau) - \zeta_2(s, e^{\pi i} \mid \tau, e^{\pi i}) \} \Big|_{s=-2N} &= \frac{\partial}{\partial s} \{ (\tau^{-s} - e^{-\pi i s}) \zeta(s) \} \Big|_{s=-2N} \\ &= (\tau^{2N} - 1) \frac{\partial \zeta}{\partial s}(-2N). \end{aligned}$$

Recall the functional equation of the Riemann zeta function.

$$\zeta(s) = 2\Gamma(1-s) \sin\left(\frac{\pi s}{2}\right) (2\pi)^{s-1} \zeta(1-s).$$

Hence, it follows that

$$\frac{\partial \zeta}{\partial s}(-2N) = \frac{(-1)^N}{2} (2N)! (2\pi)^{-2N} \zeta(2N+1).$$

Therefore,

$$\frac{\partial g}{\partial s}(-2N, \tau) = \frac{\pi i B_{2,2+2N}(0 \mid 1, \tau)}{(2N+2)(2N+1)} + \frac{(-1)^N}{2} (\tau^{2N} - 1) (2N)! (2\pi)^{-2N} \zeta(2N+1).$$

(3) : We remark that

$$\begin{aligned} \zeta_2(s, \tau \mid e^{\pi i}, \tau) - e^{\pi i s} \zeta_2(s, e^{\pi i} \mid \tau, e^{\pi i}) &= \{ \zeta_2(s, \tau \mid e^{\pi i}, \tau) - \zeta_2(s, e^{\pi i} \mid \tau, e^{\pi i}) \} \\ &\quad + (1 - e^{\pi i s}) \zeta_2(s, e^{\pi i} \mid \tau, e^{\pi i}). \end{aligned}$$

Therefore, by (5.1), (5.2) and  $\zeta(2N) = -\frac{1}{2} \frac{B_{2N}}{(2N)!} (2\pi i)^{2N}$ , we have

$$\begin{aligned} g(2N, \tau) &= \{ \zeta_2(2N, \tau \mid e^{\pi i}, \tau) - \zeta_2(2N, e^{\pi i} \mid \tau, e^{\pi i}) \} + \lim_{s \rightarrow 2N} (1 - e^{\pi i s}) \zeta_2(s, e^{\pi i} \mid \tau, e^{\pi i}) \\ &= (\tau^{-2N} - 1) \left( -\frac{1}{2} \frac{B_{2N}}{(2N)!} (2\pi i)^{2N} \right) + \frac{\pi i}{\tau} \delta_{1,N}. \end{aligned}$$

□

**Proposition 5.3.** (1) (*Inversion formula for the Dedekind  $\eta$ -function*) Let

$$(5.6) \quad \eta(\tau) := e^{\frac{\pi i}{12}\tau} \prod_{m=1}^{\infty} (1 - e^{2\pi i m \tau})$$

be the Dedekind  $\eta$ -function. We have

$$(5.7) \quad \eta\left(e^{\pi i} \frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}} \eta(\tau).$$

Here  $\sqrt{\frac{\tau}{i}}$  being that branch taking the value 1 at  $\tau = i$ .

(2) (*Ramanujan's formula*) For any  $N \in \mathbf{N}$ , we have

$$(5.8) \quad \frac{1}{2}\zeta(2N+1) + \sum_{n=1}^{\infty} \frac{1}{n^{2N+1}} \frac{e^{2\pi i n \tau}}{1 - e^{2\pi i n \tau}} = \tau^{2N} \left\{ \frac{1}{2}\zeta(2N+1) + \sum_{n=1}^{\infty} \frac{1}{n^{2N+1}} \frac{e^{-2\pi i n \frac{1}{\tau}}}{1 - e^{-2\pi i n \frac{1}{\tau}}} \right\} \\ + \frac{1}{2} \frac{(2\pi i)^{2N+1}}{(2N+2)!} B_{2,2+2N}(0 \mid \tau, 1).$$

(3) (*Inversion formula for the Eisenstein series/Lambert series*) For any  $N \in \mathbf{N}$ , we have

$$(5.9) \quad \sum_{n=1}^{\infty} \frac{n^{2N-1} e^{-2\pi i n \frac{1}{\tau}}}{1 - e^{-2\pi i n \frac{1}{\tau}}} - \frac{B_{2N}}{4N} = \tau^{2N} \left( \sum_{n=1}^{\infty} \frac{n^{2N-1} e^{2\pi i n \tau}}{1 - e^{2\pi i n \tau}} - \frac{B_{2N}}{4N} \right) - \frac{\tau}{4\pi i} \delta_{1,N}.$$

*Proof.* (1) : By Corollary 4.8 and Corollary 4.9, we have

$$\exp\left(\frac{\partial g}{\partial s}(0, \tau)\right) = \exp\left(\frac{\partial \xi_2}{\partial s}(0, \tau \mid e^{\pi i}; \tau) - \frac{\partial \xi_2}{\partial s}\left(0, e^{\pi i} \frac{1}{\tau} \mid e^{\pi i}; e^{\pi i} \frac{1}{\tau}\right)\right) = \frac{h\left(e^{\pi i} \frac{1}{\tau}\right)}{h(\tau)}.$$

Here,  $h(\tau) := \prod_{m=1}^{\infty} (1 - e^{2\pi i m \tau})$ . On the other hand, by (5.3)

$$\exp\left(\frac{\partial g}{\partial s}(0, \tau)\right) = \exp\left(-\frac{\pi i}{4} + \frac{\pi i}{12} \left(\tau + \frac{1}{\tau}\right) + \frac{1}{2} \log \tau\right).$$

Therefore,

$$e^{\frac{\pi i}{12} \frac{e^{\pi i}}{\tau}} h\left(e^{\pi i} \frac{1}{\tau}\right) = \exp\left(\frac{1}{2} \log \tau - \frac{\pi i}{4}\right) e^{\frac{\pi i}{12} \tau} h(\tau).$$

Since  $\eta(\tau) = e^{\frac{\pi i}{12} \tau} h(\tau)$ , we obtain (5.7).

(2) : By Corollary 4.8 and Corollary 4.9, we have

$$\frac{\partial g}{\partial s}(-2N, \tau) = \frac{(2N)!}{(2\pi i)^{2N}} \sum_{n=1}^{\infty} \frac{1}{n^{2N+1}} \frac{e^{2\pi i n \tau}}{1 - e^{2\pi i n \tau}} - \tau^{2N} \frac{(2N)!}{(2\pi i)^{2N}} \sum_{n=1}^{\infty} \frac{1}{n^{2N+1}} \frac{e^{-2\pi i n \frac{1}{\tau}}}{1 - e^{-2\pi i n \frac{1}{\tau}}}.$$

Hence we obtain (5.8) by (5.4).

(3) : By (4.10), we have

$$g(2N, \tau) = \frac{(2\pi i)^{2N}}{(2N-1)!} \sum_{n=1}^{\infty} \frac{n^{2N-1} e^{2\pi i n \tau}}{1 - e^{2\pi i n \tau}} - \tau^{-2N} \frac{(2\pi i)^{2N}}{(2N-1)!} \sum_{n=1}^{\infty} \frac{n^{2N-1} e^{-2\pi i n \frac{1}{\tau}}}{1 - e^{-2\pi i n \frac{1}{\tau}}}.$$

Hence we obtain (5.9) by (5.5).  $\square$

## 5.2 Fourier expansion of the Barnes zeta function

**Theorem 5.4.** *Let  $r \geq 2$ ,  $z \in D$ . We assume that  $z, \omega_1, \dots, \omega_r \in \mathfrak{H}$  satisfy the condition [ORC]. Then, for all  $s \in \mathbf{C}$ , we obtain*

$$\begin{aligned}
 (5.10) \quad \zeta_r(s, z \mid \omega) &= \frac{1}{2i \sin(\pi s)} \\
 &\cdot \left\{ \sum_{k=1}^r (-1)^{k-1} \xi_r(s, |\omega|_{[1, k-1]}^+ + e^{-\pi i} z \mid \omega_k; \widehat{\omega}^-[k, r](k)) \right. \\
 &\quad \left. - \sum_{k=1}^r (-1)^{r-k} e^{-\pi i s} \xi_r(s, z + e^{-\pi i} |\omega|_{[k+1, r]}^+ \mid \omega_k; \widehat{\omega}^-[k, r](k)) \right\} \\
 (5.11) \quad &= (2\pi)^{s-1} \Gamma(1-s) \\
 &\cdot \left\{ e^{\frac{\pi}{2}i(s-1)} \sum_{k=1}^r \omega_k^{-s} \sum_{n=1}^{\infty} n^{s-1} e^{2\pi i n z_k} \prod_{j=1, j \neq k}^r (1 - e^{2\pi i n \omega_{jk}})^{-1} \right. \\
 &\quad \left. + e^{-\frac{\pi}{2}i(s-1)} \sum_{k=1}^r \omega_k^{-s} \sum_{n=1}^{\infty} n^{s-1} e^{-2\pi i n z_k} \prod_{j=1, j \neq k}^r (1 - e^{-2\pi i n \omega_{jk}})^{-1} \right\}.
 \end{aligned}$$

*Proof.* Put

$$F(s, z \mid \omega) := f_+(s, z \mid \omega) - e^{-\pi i s} f_-(s, z \mid \omega).$$

Let  $z \in D$ . Then, by Theorem 4.11, we have

$$\begin{aligned}
 F(s, z \mid \omega) &= \sum_{k=1}^r (-1)^{k-1} \xi_r(s, |\omega|_{[1, k-1]}^+ + e^{-\pi i} z \mid \omega_k; \widehat{\omega}^-[k, r](k)) \\
 &\quad - \sum_{k=1}^r (-1)^{r-k} e^{-\pi i s} \xi_r(s, z + e^{-\pi i} |\omega|_{[k+1, r]}^+ \mid \omega_k; \widehat{\omega}^-[k, r](k)) \\
 &= \frac{(2\pi)^s}{\Gamma(s)} \left\{ e^{\frac{\pi}{2}i s} \sum_{k=1}^r \omega_k^{-s} \sum_{n=1}^{\infty} n^{s-1} e^{2\pi i n z_k} \prod_{j=1, j \neq k}^r (1 - e^{2\pi i n \omega_{jk}})^{-1} \right. \\
 &\quad \left. - e^{-\frac{\pi}{2}i s} \sum_{k=1}^r \omega_k^{-s} \sum_{n=1}^{\infty} n^{s-1} e^{-2\pi i n z_k} \prod_{j=1, j \neq k}^r (1 - e^{-2\pi i n \omega_{jk}})^{-1} \right\}.
 \end{aligned}$$

On the other hand, by the definitions of  $f_+(s, z \mid \omega)$  and  $f_-(s, z \mid \omega)$ ,

$$\begin{aligned}
 F(s, z \mid \omega) &= (e^{\pi i s} - e^{-\pi i s}) \zeta_r(s, z \mid \omega) \\
 &\quad + (-1)^{r-1} (\zeta_r(s, |\omega|^+ + e^{-\pi i} z \mid \omega) - e^{-\pi i s} \zeta_r(s, z + e^{-\pi i} |\omega|^+ \mid e^{-\pi i} \omega)).
 \end{aligned}$$

Since  $z \in D$ , one has  $e^{\pi i} (z + e^{-\pi i} |\omega|^+) = |\omega|^+ + e^{-\pi i} z$ . By Lemma 3.1 we have

$$\begin{aligned}
 e^{-\pi i s} \zeta_r(s, z + e^{-\pi i} |\omega|^+ \mid e^{-\pi i} \omega) &= \zeta_r(s, e^{\pi i} (z + e^{-\pi i} |\omega|^+) \mid \omega) \\
 &= \zeta_r(s, |\omega|^+ + e^{-\pi i} z \mid \omega).
 \end{aligned}$$

Therefore,

$$F(s, z \mid \omega) = (e^{\pi i s} - e^{-\pi i s}) \zeta_r(s, z \mid \omega) = 2i \sin(\pi s) \zeta_r(s, z \mid \omega).$$

Hence, we have (5.10). Moreover it is obvious from Theorem 4.11 that

$$\begin{aligned} \zeta_r(s, z \mid \omega) = & \frac{(2\pi)^s}{2i\Gamma(s) \sin(\pi s)} \left\{ e^{\frac{\pi}{2} i s} \sum_{k=1}^r \omega_k^{-s} \sum_{n=1}^{\infty} n^{s-1} e^{2\pi i n z_k} \prod_{j=1, j \neq k}^r (1 - e^{2\pi i n \omega_{jk}})^{-1} \right. \\ & \left. - e^{-\frac{\pi}{2} i s} \sum_{k=1}^r \omega_k^{-s} \sum_{n=1}^{\infty} n^{s-1} e^{-2\pi i n z_k} \prod_{j=1, j \neq k}^r (1 - e^{-2\pi i n \omega_{jk}})^{-1} \right\}. \end{aligned}$$

Hence (5.11) follows from the reflection formula for the Gamma function.  $\square$

**Remark 5.5.** The formula (5.11) has given by Komuri, Matsumoto and Tsumura [KMT]. In particular, when  $r = 1$ ,  $\operatorname{Re}(s) < 0$  and  $z = a\omega_1$  ( $0 < a < 1$ ), the formula (5.11) gives the following functional equation of Hurwitz's zeta function.

$$(5.12) \quad \zeta_1(s, z \mid \omega_1) = \frac{1}{2i \sin(\pi s)} \{ \xi_1(s, e^{-\pi i} z \mid \omega_1) - e^{-\pi i s} \zeta_1(s, z \mid \omega_1) \}$$

$$(5.13) \quad \begin{aligned} &= (2\pi)^{s-1} \Gamma(1-s) \omega_1^{-s} \\ &\quad \cdot \left\{ e^{\frac{\pi}{2} i (s-1)} \sum_{n=1}^{\infty} n^{s-1} e^{2\pi i n z_1} + e^{-\frac{\pi}{2} i (s-1)} \sum_{n=1}^{\infty} n^{s-1} e^{-2\pi i n z_1} \right\}. \end{aligned}$$

**Corollary 5.6.** We assume that  $r \geq 2$ ,  $z \in D$  and  $\omega_1, \dots, \omega_r$  satisfy the condition [ORC]. Then

(1) For all  $m \in \mathbf{N}_0$ ,

$$(5.14) \quad B_{r,m}(z \mid \omega) = (-1)^r (2\pi i)^{r-1-m} m! \sum_{k=1}^r \omega_k^{m-r} \sum_{n \in \mathbf{Z} \setminus \{0\}} n^{r-1-m} e^{2\pi i n z_k} \prod_{j=1, j \neq k}^r (1 - e^{2\pi i n \omega_{jk}})^{-1}.$$

(2) For all  $m \in \mathbf{N}$ ,

$$(5.15) \quad \sum_{k=1}^r \omega_k^{-(r+m)} \sum_{n \in \mathbf{Z} \setminus \{0\}} n^{r+m-1} e^{2\pi i n z_k} \prod_{j=1, j \neq k}^r (1 - e^{2\pi i n \omega_{jk}})^{-1} = 0.$$

*Proof.* (1) : We notice that the left-hand side of (5.14) is a multiple Bernoulli polynomial, which is a rational function of  $\omega_1, \dots, \omega_r$  and that the right-hand side of (5.14) converges absolutely, when  $z \in D$  and  $\arg(\omega_j) \neq \arg(\omega_k)$  ( $j \neq k$ ). Hence, by analytic continuation, it is enough to show the assertion when we assume that  $\omega_1, \dots, \omega_r$  satisfy the same conditions of Theorem 4.11.

If  $z \in D$ , by (5.11), for all  $m \in \mathbf{N}$ , we have

$$\zeta_r(1-m, z \mid \omega) = \frac{(m-1)!}{(2\pi i)^m} \sum_{k=1}^r \omega_k^{m-1} \sum_{n \in \mathbf{Z} \setminus \{0\}} \frac{e^{2\pi i n z_k}}{n^m} \prod_{j=1, j \neq k}^r (1 - e^{2\pi i n \omega_{jk}})^{-1}.$$

Using (3.15), we have

$$B_{r,r+m-1}(z \mid \omega) = (-1)^r \frac{(m+r-1)!}{(2\pi i)^m} \sum_{k=1}^r \omega_k^{m-1} \sum_{n \in \mathbf{Z} \setminus \{0\}} \frac{e^{2\pi i n z_k}}{n^m} \prod_{j=1, j \neq k}^r (1 - e^{2\pi i n \omega_{jk}})^{-1}.$$

Replacing  $m$  with  $m - r + 1$ , for all  $m \in \mathbf{Z}_{\geq r}$ , we obtain (5.14). On the other hand, by (5.11), if  $z \in D$ , we see that

$$\begin{aligned} \operatorname{Res}_{s=m} \zeta_r(s, z \mid \omega) ds &= \lim_{s \rightarrow m} (s - m) \zeta_r(s, z \mid \omega) \\ &= \frac{(-1)^m (2\pi i)^{m-1}}{(m-1)!} \sum_{k=1}^r \omega_k^{-m} \sum_{n \in \mathbf{Z} \setminus \{0\}} n^{m-1} e^{2\pi i n z_k} \prod_{j=1, j \neq k}^r (1 - e^{2\pi i n \omega_{jk}})^{-1} \end{aligned}$$

for all  $m = 1, \dots, r$ . Hence it follows from (3.16) that

$$B_{r,r-m}(z \mid \omega) = (-1)^r (2\pi i)^{m-1} (r-m)! \sum_{k=1}^r \omega_k^{-m} \sum_{n \in \mathbf{Z} \setminus \{0\}} n^{m-1} e^{2\pi i n z_k} \prod_{j=1, j \neq k}^r (1 - e^{2\pi i n \omega_{jk}})^{-1}.$$

Replacing  $m$  with  $r - m$ , we obtain the result.

(2) : Since  $\zeta_r(s, z \mid \omega)$  is holomorphic when  $\operatorname{Re}(s) > r$ , we have

$$\lim_{s \rightarrow m} \frac{\zeta_r(s + r, z \mid \omega)}{\Gamma(1 - r - s)} = 0 \quad (m \in \mathbf{N}).$$

Since  $\omega_1, \dots, \omega_r$  satisfy the same conditions of Theorem 4.11, if  $z \in D$ , it follows that

$$\lim_{s \rightarrow m} \frac{\zeta_r(s + r, z \mid \omega)}{\Gamma(1 - r - s)} = (2\pi i)^{r+m-1} \sum_{k=1}^r \omega_k^{-(r+m)} \sum_{n \in \mathbf{Z} \setminus \{0\}} n^{r+m-1} e^{2\pi i n z_k} \prod_{j=1, j \neq k}^r (1 - e^{2\pi i n \omega_{jk}})^{-1}.$$

Hence (5.15) follows immediately. □

**Remark 5.7.** Suppose  $r = 1$ . If  $z = a\omega_1$  ( $0 < a < 1$ ), for all  $m \in \mathbf{N}$ , one has

$$B_{1,m}(z \mid \omega_1) = -\frac{m!}{(2\pi i)^m} \omega_1^{m-1} \sum_{n \in \mathbf{Z} \setminus \{0\}} \frac{e^{2\pi i n z_1}}{n^m}.$$

Here, when  $m = 1$ , the summation on  $n$  is meant

$$\lim_{N \rightarrow \infty} \sum_{n=-N, n \neq 0}^N \frac{e^{2\pi i n z_1}}{n}.$$

### 5.3 Multiple Iseki's formulas

We prove the following multiple-analogue of Iseki's formula.

**Theorem 5.8.** *We assume that  $r \geq 2$ ,  $z \in D$  and  $\omega_1, \dots, \omega_r$  satisfy the condition [ORC]. Put*

$$(5.16) \quad f(s, z \mid \omega) := \zeta_r(s, z \mid \omega) + (-1)^{r-1} \zeta_r(s, |\omega|^+ + e^{-\pi i} z \mid \omega).$$

Then, for all  $N \in \mathbf{N}_0$ , we have

$$(5.17) \quad \begin{aligned} \frac{\partial f}{\partial s}(-2N, z \mid \omega) &= (-1)^{r+1} \pi i \frac{(2N)!}{(2N+r)!} B_{r,r+2N}(z \mid \omega) \\ &\quad + \frac{(-1)^N (2N)!}{(2\pi)^{2N}} \sum_{k=1}^r \omega_k^{2N} \sum_{n=1}^{\infty} \frac{e^{2\pi i n z_k}}{n^{2N+1}} \prod_{j=1, j \neq k}^r (1 - e^{2\pi i n \omega_{jk}})^{-1} \\ &= (-1)^r \pi i \frac{(2N)!}{(2N+r)!} B_{r,r+2N}(z \mid \omega) \\ &\quad + \frac{(-1)^N (2N)!}{(2\pi)^{2N}} \sum_{k=1}^r \omega_k^{2N} \sum_{n=1}^{\infty} \frac{e^{-2\pi i n z_k}}{n^{2N+1}} \prod_{j=1, j \neq k}^r (1 - e^{-2\pi i n \omega_{jk}})^{-1}. \end{aligned}$$

In particular, for all  $z \in \mathbf{C}$ , we have

$$(5.18) \quad \begin{aligned} \exp\left(-\frac{\partial f}{\partial s}(0, z \mid \omega)\right) &= \exp\left(\frac{(-1)^r \pi i}{r!} B_{r,r}(z \mid \omega)\right) \prod_{k=1}^r \widetilde{(x_k; \widehat{\mathbf{q}}_k)}_{r-1, \infty} \\ &= \exp\left(\frac{(-1)^{r-1} \pi i}{r!} B_{r,r}(z \mid \omega)\right) \prod_{k=1}^r \widetilde{(x_k^{-1}; \widehat{\mathbf{q}}_k^{-1})}_{r-1, \infty}. \end{aligned}$$

*Proof.* If  $z \in D$ , then

$$f(s, z \mid \omega) = e^{-\pi i s} \zeta_r(s, e^{-\pi i} z \mid e^{-\pi i} \omega) + (-1)^{r-1} \zeta_r(s, |\omega|^+ + e^{-\pi i} z \mid \omega).$$

Thus, by (3.15), Lemma 3.5 and (4.24),

$$\begin{aligned} \frac{\partial f}{\partial s}(-2N, z \mid \omega) &= -\pi i \zeta_r(-2N, e^{-\pi i} z \mid e^{-\pi i} \omega) + \frac{\partial f_+}{\partial s}(-2N, z \mid \omega) \\ &= (-1)^{r+1} \pi i \frac{(2N)!}{(2N+r)!} B_{r,r+2N}(z \mid \omega) \\ &\quad + \frac{(-1)^N (2N)!}{(2\pi)^{2N}} \sum_{k=1}^r \omega_k^{2N} \sum_{n=1}^{\infty} \frac{e^{2\pi i n z_k}}{n^{2N+1}} \prod_{j=1, j \neq k}^r (1 - e^{2\pi i n \omega_{jk}})^{-1}. \end{aligned}$$

On the other hand, since  $z \in D$ , we see that  $e^{-\pi i}(|\omega|^+ + e^{-\pi i} z) = z + e^{-\pi i} |\omega|^+$ . Therefore,

$$f(s, z \mid \omega) = \zeta_r(s, z \mid \omega) + (-1)^{r-1} e^{-\pi i s} \zeta_r(s, z + e^{-\pi i} |\omega|^+ \mid e^{-\pi i} \omega).$$

Hence, by (3.15), Lemma 3.5 and (4.24),

$$\begin{aligned} \frac{\partial f}{\partial s}(-2N, z \mid \omega) &= -\pi i(-1)^{r-1} \zeta_r(-2N, z + e^{-\pi i} |\omega|^+ \mid e^{-\pi i} \omega) + \frac{\partial f_-}{\partial s}(-2N, z \mid \omega) \\ &= (-1)^r \pi i \frac{(2N)!}{(2N+r)!} B_{r,r+2N}(z \mid \omega) \\ &\quad + \frac{(-1)^N (2N)!}{(2\pi)^{2N}} \sum_{k=1}^r \omega_k^{2N} \sum_{n=1}^{\infty} \frac{e^{-2\pi i n z_k}}{n^{2N+1}} \prod_{j=1, j \neq k}^r (1 - e^{-2\pi i n \omega_{jk}})^{-1}. \end{aligned}$$

Consequently we obtain (5.17). Similarly,

$$\begin{aligned} \exp \left( -\frac{\partial f}{\partial s}(0, z \mid \omega) \right) &= \exp \left( \frac{(-1)^r \pi i}{r!} B_{r,r}(z \mid \omega) \right) \exp \left( -\frac{\partial f_+}{\partial s}(0, z \mid \omega) \right) \\ &= \exp \left( \frac{(-1)^{r-1} \pi i}{r!} B_{r,r}(z \mid \omega) \right) \exp \left( -\frac{\partial f_-}{\partial s}(0, z \mid \omega) \right). \end{aligned}$$

Hence the result (5.18) follows from (4.25) by analytic continuation.  $\square$

**Remark 5.9.** (1) For  $r = 1$ , we have

$$\begin{aligned} (5.19) \quad \frac{\partial f}{\partial s}(-2N, z \mid \omega_1) &= \frac{\pi i}{2N+1} B_{1,2N+1}(z \mid \omega_1) + \frac{(-1)^N (2N)!}{(2\pi)^{2N}} \omega_1^{2N} \sum_{n=1}^{\infty} \frac{e^{2\pi i n a}}{n^{2N+1}} \\ &= -\frac{\pi i}{2N+1} B_{1,2N+1}(z \mid \omega_1) + \frac{(-1)^N (2N)!}{(2\pi)^{2N}} \omega_1^{2N} \sum_{n=1}^{\infty} \frac{e^{-2\pi i n a}}{n^{2N+1}}, \end{aligned}$$

where  $z = a\omega_1$  ( $0 < a < 1$ ). In particular, for all  $z \in \mathbf{C}$ ,

$$(5.20) \quad \exp \left( -\frac{\partial f}{\partial s}(0, z \mid \omega) \right) = e^{-\pi i \left( \frac{z}{\omega_1} - \frac{1}{2} \right)} (1 - e^{2\pi i a}) = e^{\pi i \left( \frac{z}{\omega_1} - \frac{1}{2} \right)} (1 - e^{-2\pi i a}).$$

(2) We remark that Narukawa has proved (5.18) in [Na] and the left hand side of (5.18) is a multiple sine function.

(3) If  $r = 2$ , (5.18) gives Iseki's formula [I].

(4) By Remark 4.16, if  $z \in D$ , the second equality of (5.17) and that of (5.18) are true for replacing the condition [ORC] by  $\arg(\omega_j) \neq \arg(\omega_k)$  ( $j \neq k$ ).

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